1 Quiz 10 (Apr 15) solutions

• 1. Give the appropriate sample spaces for each of the following experiments: (a) outcomes when three coins are flipped, (b) the genotype of an offspring when one parent is heterozygous and the other is homozygous recessive.

(a) The sample space for three coin flips (with H for heads, T for tails) is
$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

(b) Note that if one parent is heterozygous it has genotype Aa for a dominant allele A, recessive allele a, and if the other is homozygous recessive it has genotype aa. Then the sample space of genotypes of offspring is $\{Aa, aa\}$ since the Punnett square looks like

$$\begin{array}{cc} & \begin{array}{cc} A & a \end{array} \\ \begin{array}{c} a \\ a \end{array} & \begin{pmatrix} Aa & aa \\ Aa & aa \end{pmatrix} \end{array}$$

• 2. Consider the experiment where two dice are rolled. Find the probability of the following events: (a) "doubles", (b) "sum of 5".

(a) Notice there are 6 possibilities for doubles, you either roll a double 1, double 2, ... , or a double 6. The sample space for two dice being rolled has size $6 * 6 = 36$, so the probability is $\frac{6}{36} = \frac{1}{6}$.

(b) Notice that there are only 4 possibilities for a sum of 5: $\{\{1, 4\}, \{2, 3\}, \{3, 2\}, \{4, 1\}\}$. Thus the probability is $\frac{4}{36} = \frac{1}{9}$.

2 WS 10 (Apr 15) solutions

• 1. A mother and father each have blood type AB. Find the probability that their child has the following blood type: (a) AB, (b) A, (c) B.

We draw out the Punnet square:

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} AA & AB \\ AB & BB \end{pmatrix} \end{array}$$

(a) The probability of AB is $\frac{2}{4} = \frac{1}{2}$.

(b) The probability of type A (AA) is $\frac{1}{4}$.

(c) The probability of type B (BB) is $\frac{1}{4}$.

• 2. Suppose that a family has five children. Find the probability that there are exactly three girls and two boys.

We could use the formula $\binom{n}{k}p^k(1-p)^{n-k}$ (for exactly k successes out of n trials of a binomial experiment) or count the possibilities:
$\{BBGGG, BGBGG, BGGBG, BGGGB, GBBGG, GBGBG, GBGGB, GGBBG, GGBGB, GGGBB\}$ or 10 possibilities out of $2^5 = 32$ (two choices boy or girl for each of the 5 children). Thus the probability is $\binom{5}{2}(\frac{1}{2})^2 * (1 - \frac{1}{2})^3 = \frac{10}{2^5} = \frac{10}{32} = 0.3125$.

• 3. One of the genes for eye color in humans has two alleles, B and b. The brown allele, B, is dominant over the blue allele, b, and so genotypes BB and Bb have brown eyes, while genotype bb has blue eyes. Suppose that a child has brown eyes and the child's father has blue eyes.

(a) What is the genotype of the child?

Since the child has brown eyes, it has either BB or Bb genotype, but the father has type bb, so the child is guaranteed a b allele. Thus the child has type Bb.

(b) What are the possible genotypes of the mother?

The mother must have brown eyes since the father has blue (doubly recessive): thus she can have BB or Bb genotypes.

(c) Suppose that the child grows up and mates with a blue-eyed individual. What is the probability that their offspring will have brown eyes?

Since the child has genotype Bb and mates with a bb genotype, the Punnett square is as follows:

$$\begin{array}{cc} & \begin{array}{cc} B & b \end{array} \\ \begin{array}{c} b \\ b \end{array} & \begin{pmatrix} Bb & bb \\ Bb & bb \end{pmatrix} \end{array}$$

Thus the probability that the child has brown eyes is $\frac{2}{4} = \frac{1}{2}$.

3 Quiz 11 (Apr 22) solutions

• 1. Using only four genes denoted by (N-P-M-G), how many different gene orders are there?

There are $4 * 3 * 2 * 1 = 24$ possible gene orders, four choices for the first gene, three for the second (after choosing the first), 2 for the next, and only 1 choice for the last.

• 2. Suppose that you want to plant four trees in a plot and can choose from 10 different species. How many ways can be chosen to plant the trees in a plot?

We will be treating (T1, T2, T3, T4) as equal to (T2, T1, T3, T4) so that order doesn't matter (planting an oak and a spruce is the same as planting a spruce and an oak). This tells us we're dealing with combinations. Since we're counting how many ways we can choose 4 trees out of 10, thus there are $\binom{10}{4} = \frac{10!}{4! * (10-4)!} = \frac{10*9*8*...*2*1}{4*3*2*1*6*5*4*3*2*1} = 210$.

• 3. Suppose that a family has five children. Find the probability that there are four girls and one boy.

We can either go straight to the formula (for exactly k successes out of n trials of a binomial experiment) $\binom{n}{k} p^k (1-p)^{n-k}$ (with $n = 5, k = 1, p = \frac{1}{2}$) or we can go by counting the possibilities $\{BGGGG, GBGGG, GGBGG, GGGBG, G$ thus our probability is $5/2^5$.

• 4. A coin is tossed three times. Let B be the event "at least one tail" and C the event "a head on the third toss". List all elementary events in each of the following events: (a) B, (b) $B \cap C$, (c) $C - B$.

(a) B

Since event B is having at least one tail, $B = \{HHT, THH, HTH, TTH, THT, HTT, TTT\}$.

(b) $B \cap C$

Note $B \cap C$ tells us to choose elementary events in both B and C. Thus we need at least one tail AND a head on the third toss: $B \cap C = \{THH, HTH, TTH\}$.

(c) $C - B$

The set $C - B$ consists of events that are in C but aren't in B. Since $C = \{HHH, THH, TTH, HTH\}$, but THH, TTH, HTH are all in B, $C - B = \{HHH\}$.

4 WS 11 (Apr 22) solutions

• 1. Suppose that there are five types of toppings to put on a certain pizza. If you want to have only two toppings, how many choices do you have?

We'll be treating a mushroom and pepperoni pizza to be the same as a pepperoni and mushroom pizza, so order of toppings doesn't matter. This tells us we're dealing with combinations: $\binom{5}{2} = 10$.

• 2. Consider the following using the sample space $\{1, 2, 3, 4, 5, 6\}$ and events $A = \{1, 3, 5\}$ and $B = \{1, 2, 3, 4\}$. (a) Find $A \cup B$ and $A \cap B$, (b) find $\bar{A}$, (c) find $A \bar{\cup} B$, (d) are A and B mutually exclusive?

(a) Find $A \cup B$ and $A \cap B$

Note $A \cup B$ consists of elementary events that are in A OR B: $A \cup B = \{1, 2, 3, 4, 5\}$, while $A \cap B = \{1, 3\}$ is the set of common elementary events.

(b) find $\bar{A}$

Note $\bar{A}$ is the set of elementary events of the sample space that aren't in A: $\bar{A} = \{2, 4, 6\}$.

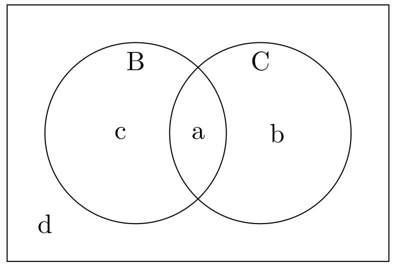
(c) find $A \bar{\cup} B$

Note $A \bar{\cup} B$ is the set of elementary events of the sample space that aren't in $A \cup B$: $A \bar{\cup} B = \{6\}$.

(d) are A and B mutually exclusive?

The events A and B are not mutually exclusive because $A \cap B$ is nonempty: $A \cap B = \{1, 3\}$.

• 3. In the population of Dry Gulch, 75% of the population are cowboys and 90% are beer drinkers. Only 5% stay away from beer and horses. (a) Suppose that C is the event "cowboy" and that B is the event "beer drinker". Express, in words, the events a,b,c, and d, (b) find the proportion of the population that are beer drinking cowboys.



(a) Well, the elementary event a is a member of the population who is both a cowboy and a beer drinker. The event b is a member that is a cowboy but not a beer drinker, and c is a member that is a beer drinker but not a cowboy. The event d is a member of Dry Gulch that is not a cowboy nor a beer drinker.

(b) To find the proportion of the population that are beer-drinking cowboys, note that $P(B) = 0.9$, $P(C) = 0.75$, $P(\bar{B} \cap \bar{C}) = 0.05$ by the problem statement. We need to find $P(B \cap C)$, i.e. the proportion of the population that are cowboys and are beer drinkers.

By properties of union and intersection (above example 11.9 in the textbook), $P(B \cap C) = P(B) + P(C) - P(B \cup C)$, and $P(\bar{B} \cap \bar{C}) = 1 - P(B \cup C) = 0.05$ (think of this as saying, if something is not in B AND not in C, then it is not in B OR C, so $\bar{B} \cap \bar{C} = \overline{B \cup C}$). This means $P(B \cup C) = 0.95$, so $P(B \cap C) = P(B) + P(C) - P(B \cup C) = 0.9 + 0.75 - 0.95 = 0.7$. Thus 70% of the population are both cowboys and beer drinkers.

5 Quiz 12 (Apr 29) solutions

• 1. Let A and B be two independent events with $P(A) = P(B) = 0.5$. Find $P(A \cap B)$.

Note that for independent events, $P(A \cap B) = P(A) * P(B)$, so $P(A \cap B) = 0.5 * 0.5 = 0.25$.

• 2. Assume that a couple is equally likely to have a boy or a girl. A family has three children. Find the probability of the following events: (a) "all children are girls", (b) "at least one boy", (c) "at least two girls".

(a) "all children are girls"

Since the couple is equally likely to have a boy or a girl, P(G)=0.5, and the events are independent, $P(allgirls) = P(1stchildisagirl) * P(2ndchildisagirl) * P(3rdchildisagirl) = (0.5)^3 = .125$.

Another way is to note that $\{GGG\}$ is the one possibility out of the total sample space which is of size $2^3 = 8$, so the probability is $\frac{1}{8}$.

(b) "at least one boy"

We can use the fact that having at least one boy is the opposite of having all girls. Thus $P(\geq 1boy) = 1 - P(allgirls) = 1 - \frac{1}{8} = \frac{7}{8} = 1 - 0.125 = 0.875$.

Another way is to add the probabilities of having exactly 1 boy, then exactly 2 boys, then 3 boys. Since

$P(1boy) = 3/8$, $P(2boys) = 3/8$, $P(3boys) = 1/8$ (either by counting possibilities or using the formula $\binom{n}{k}p^k(1-p)^{n-k}$) we add and again obtain 7/8.

(c) "at least two girls"

Here we could just add $P(2girls) + P(3girls) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = 0.5$.

• 3. For two events A and C given that $P(A|C) = 0.7$, $P(A) = 0.3$, and $P(C) = 0.1$, find $P(C|A)$.

By Bayes' theorem, $P(C|A) = \frac{P(A|C)*P(C)}{P(A)} = \frac{0.7*0.1}{0.3} = \frac{7}{30} = 0.233...$

• (Bonus) A probability experiment has four possible outcomes: e_1, e_2, e_3, e_4. The outcome of e_1 is four times as likely as each of the three remaining outcomes. Find the probability of e_1.

Note that $P(e_1) = 4 * P(e_2) = 4 * P(e_3) = 4 * P(e_4)$ implies $P(e_2) = P(e_3) = P(e_4)$, and $P(e_1) + P(e_2) + P(e_3) + P(e_4) = 1$. Thus, replacing $P(e_1)$ with $4 * P(e_2)$ and $P(e_3)$, $P(e_4)$ with $P(e_2)$, we have $4 * P(e_2) + P(e_2) + P(e_2) + P(e_2) = 1 = 7 * P(e_2)$. Thus $P(e_2) = \frac{1}{7}$, so $P(e_1) = \frac{4}{7}$.

6 WS 12 (Apr 29) solutions

• 1. For the two events A and B, with $P(A) = 0.8, P(B|A) = 0.3$, and $P(B|\bar{A}) = 0.4$, find the following: (a) $P(A|B)$, (b) $P(\bar{A}|B)$.

(a) $P(A|B)$

By Bayes' theorem, $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0.3*0.8}{P(B)}$. We need to find $P(B)$. We can find $P(B\cap A)$ and $P(B\cap\bar{A})$ with $P(B|A) = 0.3$, $P(B|\bar{A}) = 0.4$, and then use the fact that $P(B) = P(B\cap A) + P(B\cap\bar{A})$ with the partition theorem (since A and $\bar{A}$ form a partition of the sample space).

Note that $P(B|\bar{A}) = \frac{P(B\cap\bar{A})}{P(\bar{A})} = 0.4$, and $P(\bar{A}) = 1 - P(A) = 1 - 0.8 = 0.2$, so $P(B\cap\bar{A}) = 0.2 * 0.4 = 0.08$. Also, $P(A\cap B) = P(A)*P(B|A) = 0.8*0.3 = 0.24$. Thus $P(B) = P(B\cap A) + P(B\cap\bar{A}) = 0.24+0.08 = 0.32$.

Then $P(A|B) = \frac{0.3*0.8}{P(B)} = \frac{0.3*0.8}{0.32} = 0.75$.

(b) $P(\bar{A}|B)$

We can use the fact that $P(\bar{A}|B) + P(A|B) = 1$ (we just found $P(A|B)$ in (a)) since any element of B is either in A or it is not in A. Thus $P(\bar{A}|B) = 1 - P(A|B) = 1 - 0.75 = 0.25$.

• 2. If two fair dice are rolled, find the probabilities of the following results: (a) A sum of 10, given that the sum is greater than 5, (b) a "double" given that the sum is 12, (c) a double given that the sum is 11.

(a) A sum of 10, given that the sum is greater than 5:

If A is the event "sum of 10" and B is the event "the sum is greater than 5", we are calculating $P(A|B) = \frac{P(A\cap B)}{P(B)}$, so we find $P(A\cap B)$, then $P(B)$. A sum of 10 is already greater than 5, so $A\cap B = A$ (A is contained in B).

The elements of the sample space with a sum of 10 are $\{\{4,6\},\{5,5\},\{6,4\}\}$, so the probability of a sum of 10 (and the sum is greater than 5 trivially) is $3/36 = P(A\cap B)$.

To find $P(B)$, we remove elements from the sample space with a sum 5 or less. Thus the following set is removed:

$$\{\{1,1\},\{1,2\}\{1,3\},\{1,4\},\{2,1\},\{3,1\},\{4,1\},\{2,2\},\{2,3\},\{3,2\}\}$$

With this set removed from the space, we lose 10 elements, and the remaining 26 elements of the space will have sum greater than 5. Thus $P(B) = \frac{36-10}{36} = \frac{26}{36}$.

Thus $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{3/36}{26/36} = 3/26 = 0.115$.

(b) A double given that the sum is 12:

If we rolled a 12, we must have rolled a double six, so we always had a double. Thus $P(A|B) = 1$.

(c) A double given that the sum is 11:

If A is the event "double" and B is the event "the sum is 11", we are calculating $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Note that a double has an even sum since we end up with 2n if we roll a double n. Since 11 is odd, we can never have a double given that the sum is 11. Since $A \cap B = \emptyset$, $P(A\cap B) = 0$, so $P(A|B) = 0$.

• 3. For a test of a certain disease, given that the sensitivity of that test is 0.8 and the specificity of that test is 0.3, find the probability that the test is positive, given that the disease is not present.

We need to find $P(+|\bar{D})$. However, note that we have the specificity of the test: $P(-|\bar{D}) = 0.3$ (sensitivity is $P(+|D)$ but we don't need it here). Note that $P(+|\bar{D}) + P(-|\bar{D}) = 1$ since anyone taking the test is guaranteed either a positive or a negative result. Thus $P(+|\bar{D}) = 1 - P(-|\bar{D}) = 1 - 0.3 = 0.7$.

7 Quiz 13 (May 6) solutions

• 1. Assume that 20% of a very common insect species in your study area is parasitized. Assume that insects are parasitized independently of each other. If you collect 10 specimens of this species, what is the probability that at least one specimen in your sample is parasitized?

We calculate $1 - P(AllNonparasitized)$ since all of them being nonparasitized is the opposite of at least one of them being parasitized. The probability that one of them is nonparasitized is $1-0.2 = 0.8$, so the probability that all of them being nonparasitized is $(0.8)^10 \approx .107$. Thus $1 - P(AllNonparasitized) = 1 - .107 = .893$.

Another way to do it is to add up the probability of exactly 1 success (1 specimen parazitized) + probability of exactly 2 successes + ... + probability of exactly 10 successes (using the formula for exactly k successes out of n trials of a binomial experiment: $\binom{n}{k}p^k(1-p)^{n-k}$ (with $n = 10, 1 \le k \le 10, p = 0.2$)

• 2. For three events A, B, and C, let B and C form a partition of the sample space. If $P(A|B) = 0.7$, $P(A|C) = 0.3$, $P(B) = 0.8$, and $P(C) = 0.2$, find $P(A)$.

We will use the partition theorem since B and C form a partition of the sample space, so $P(A) = P(A|B) * P(B) + P(A|C) * P(C) = 0.7 * 0.8 + 0.3 * 0.2 = 0.52$.

• 3. In a certain area, ten percent of the rabbits in a population are slow and their chances of being captured by a fox are 0.6. Among faster rabbits, there is a 20% chance of being caught. Find the probability that a rabbit will be caught.

Call S the event of "the rabbit is slow", and C the event "the rabbit will be caught". We need to find $P(C)$. We again use the partition theorem since S, $\bar{S}$ form a partition of the sample space. Thus $P(C) = P(C|S) * P(S) + P(C|\bar{S}) * P(\bar{S})$. Note that $P(S) = 0.1, P(\bar{S}) = 1 - 0.1 = 0.9, P(C|S) = 0.6, P(C|\bar{S}) = 0.2$ by the problem statement. Thus $P(C) = 0.6 * 0.1 + 0.2 * 0.9 = 0.24$.

• (Bonus) For mutually exclusive events R_1, R_2, and R_3, we have $P(R_1) = 0.05$, $P(R_2) = 0.6$, and $P(R_3) = 0.35$. Also $P(Q|R_1) = 0.4$, $P(Q|R_2) = 0.3$, $P(Q|R_3) = 0.6$. Find $P(R_3|Q)$.

To find $P(R_3|Q) = \frac{P(Q|R_3)*P(R_3)}{P(Q)} = \frac{0.6*0.35}{P(Q)}$ by Bayes' theorem. We will find $P(Q)$. Note that R_1, R_2 and R_3 form a partition of the sample space since they are mutually exclusive events with probability summing to 1. Then by the partition theorem, $P(Q) = P(Q|R_1)*P(R_1)+P(Q|R_2)*P(R_2)+P(Q|R_3)*P(R_3) = 0.4*0.05+0.3*0.6+0.6*0.35 = 0.41$. Thus $P(R_3|Q) = \frac{0.6*0.35}{P(Q)} = \frac{0.6*0.35}{0.41} = 0.512$.

8 WS 13 (May 6) solutions

• 1. For a test for a certain disease, given that the sensitivity of that test is 0.9 and the specificity of that test is 0.5, find the probability of a false negative.

We need to find $P(-|D)$, knowing $P(+|D) = 0.9$, $P(-|\bar{D}) = 0.5$. Note that $P(-|D) + P(+|D) = 1$ since a diseased person taking one of the tests is guaranteed a positive or a negative. Then $P(-|D) = 1 - P(+|D) = 1 - 0.9 = 0.1$ is the probability of a false negative.

• 2. Suppose that you have a batch of red-flowering pea plants of which 40% are of genotype cc and 60% of genotype Cc with the C allele for red-flowers being dominant. You pick one plant at random and cross it with a white flowering pea plant of genotype cc. Find the probability that the offspring of this cross will have white flowers.

Let A denote the event "the offspring has white flowers", R denote the event "the parent has red flowers", and W the event "the parent has white flowers". Note that the events R, W partition the sample space, so by the partition theorem, $P(A) = P(A|R) * P(R) + P(A|W) * P(W)$. We need to find $P(A|R)$, the probability that the offspring has white flowers given that its parent was red-flowering (of genotype Cc). The Punnett square is as follows:

$$\begin{array}{cc} & \begin{array}{cc} C & c \end{array} \\ \begin{array}{c} c \\ c \end{array} & \begin{pmatrix} Cc & cc \\ Cc & cc \end{pmatrix} \end{array}$$

Thus $P(A|R) = 0.5$ since two of the outcomes out of four yield an offspring with white flowers (genotype cc).

Note that $P(A|W) = 1$ since the Punnett square is:

$$\begin{array}{cc} & \begin{array}{cc} c & c \end{array} \\ \begin{array}{c} c \\ c \end{array} & \begin{pmatrix} cc & cc \\ cc & cc \end{pmatrix} \end{array}$$

(a parent with white flowers has genotype cc, being crossed with a white flowering pea plant, which also has genotype cc). Thus $P(A) = P(A|R) * P(R) + P(A|W) * P(W) = 0.5 * 0.4 + 0.6 = 0.8$.

• 3. Assume that the probability that an insect of a certain species lives more than 5 days is 0.1. Find the probability that in a sample of 10 insects of this species, at least one insect will be alive after 5 days.

We calculate $1 - P(AllLive \leq 5days)$ since that is the opposite of at least one of the insects living longer than 5 days. The probability that one of them dies in 5 days or less is $1 - 0.1 = 0.9$, so the probability of all of them dying is $(0.9)^{10} \approx .349$. Thus $1 - P(AllLive \leq 5days) = 1 - .349 = 0.651$.

Again, we could go through and find $P(exactly 1 survives) + P(exactly 2 survive) + ... + P(exaclty 10 survive)$ with the formula $\binom{n}{k} p^k (1-p)^{n-k}$ (with $n = 10, 1 \leq k \leq 10, p = 0.1$) but that's much more work.